HOW TO IMMUNIZE A DEFAULTABLE BOND PORTFOLIO?


Abstract

This paper presents a new definition of a risk-adjusted duration measure of a defaultable noncallable bond portfolio, which becomes the Fischer and Weil duration in some cases. Consequently, a new immunization strategy including terms for default probabilities and default payoffs in each period as well as for a delay between the occurrence of default and the final default payoff, is derived. This approach refers to Fong and Vasicek (1984), Nawalkha and Chambers (1996), Balbás and Ibáñez (1998), and Balbás et al. (2002) studies among others but is developed under the assumption of multiple shocks in the term structure of interest rates.

Keywords: Defaultable bond portfolio; Immunization; Duration; M-Absolute

Jel classification: G11; G19

Introduction

There are various studies depicting immunization for default-free bond portfolios while recently, there has been rapid growth in the market for defaultable bonds and the demand for this research is getting increased. The present paper deals with a theoretical

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framework for the adjustment of duration for the risk of default and, consequently, with a new immunization strategy for a defaultable noncallable bond portfolio. In the literature, all the papers propose traditional models generalised to random cases. Bierwag and Kaufman (1988) develop an expression for default-adjusted duration for different patterns of expected defaults assuming a flat term structure. Fooladi et al. (1997) derive a general formula for duration in the presence of default risk based on Jonkhart’s term structure model. Jacoby (2003) extends the Fooladi et al. (1997) result and presents a model for the valuation of coupon-bearing corporate bonds. We bring together terms for default probabilities and default payoffs in each period as well as for a delay between the occurrence of default and the final default payoff but apply the approach referring to Fong and Vasicek (1984), Nawalkha and Chambers (1996), Balbás and Ibáñez (1998), and Balbás et al. (2002) studies among others. See also Kaluszka and Kondratiuk-Janyska (2004ab). Therefore we obtain a new definition of portfolio duration.To be more consistent with the observed term structure of bond prices than in the traditional approaches, (e.g. Fong and Vasicek (1984), Nawalkha and Chambers (1996), Balbás and Ibáñez (1998), and Balbás et al. (2002) and others) we model the term structure of interest rates as a random field (see e.g. Kimmel (2002)) where the supremum of absolute value of the shock deviations in the term structure from a stochastic process is no more than $\lambda$. Moreover, the innovation consists in considering multiple shocks in the term structure of interest rates but like Fooladi et al. (1997) and Jacoby (2003) it is assumed that the probability of default and the riskless term structure are statistically independent. Given the above assumptions, we propose a new definition of a risk-adjusted duration measure for a bond portfolio which one of the special cases is the Fischer and Weil duration. Moreover, a risk-adjusted dispersion measure, M-Absolute is given. It becomes the risk measure defined by Nawalkha and Chambers (1996) for default-free bond portfolio. Furthermore, we focus on immunizing the expected value of a portfolio and as a result a lower bound of the expected
portfolio return is derived. It depends upon the risk-adjusted duration and the modified M-Absolute measure among others and determines a risk minimizing strategy for defaultable bond portfolio immunization.

**Preliminary notations.**

Denote by \([0,T]\) the time interval with \(t=0\) the present moment. Let \(m\) be the investor planning horizon, \(0<m<T\). We write \(q=(q_1,q_2,\ldots,q_n)\) for the investor portfolio consisting of default-free bonds or/and bonds with default risk. This vector gives us the number of \(i\)th bond units \(q_i\) that the investor bought at time 0. The coupons paid before \(m\) will be reinvested by purchasing strip bonds (for simplicity; if the coupons were rolled over, the model can be modified). We assume \(q_i \geq 0\) to exclude short position from the analysis. Define time \(t\) as the end of period \(t\). Let \(c_{it} \geq 0\) denote the payment of \(i\)th bond due at time \(t\). Given that the firm has survived the foregoing \(t-1\) periods without default, there is a probability \(p_{it}, 0 \leq p_{it} \leq 1\), that the firm will survive period \(t\) and be able to pay \(c_{it}\). Consequently, \(1-p_{it}\) represents the probability that the firm may default during period \(t\), failing to pay at time \(t\). Then, there will be a payment of \(i\)th bond \(F_{it}(s(t))\) at time \(t+s(t)\), where \(s(t)\) represents the number of years it takes to receive final settlement from default, \(s(t) \geq 0, t+s(t) \leq T\). By \(f(t,s)\) we mean the instantaneous forward rate over the time interval \([t,s]\) and we can write that investing \$1 at time \(t\) in a zero-coupon bond we get \(\exp\left(\int_t^s f(t,u)du\right)\) at time \(s\). Hence

- \(c_{it}^{(m)} = c_{it} \exp\left(\int_t^m f(0,u)du\right)\) is the time-\(m\) present value of \(i\)th bond in the absence of default risk,
• $F^{(m)}_{it}(s(t)) = F_{it}(s(t)) \exp \left( \int_{t+s(t)}^{m} f(0,u) du \right)$ is the time-$m$ present value of $i$th bond in the presence of default risk,

• $k(t,s) = \int_{t}^{s} [f(t \land s,u) - f(0,u)] du$ is a shock in the instantaneous forward rate and $a \land b = \min(a,b)$.

While duration has been the subject of intense study, all models have been considered under the assumption of a single shock in the term structure before the moment when first payment is made. In practice, the term structure changes more often. Therefore we can write, from the bond investor’s standpoint, that the time-$m$ present value of $i$th bond is expectations formed with respect to both, statistically independent default and term structure uncertainties, and is equal to

$$V_i(k) = \mathbb{E} \sum_{t} c_{it} \exp \left( \int_{t}^{m} f(t \land m,u) du \right) p_{it}$$

$$+ F_{it}(s(t)) \exp \left( \int_{t+s(t)}^{m} f((t+s(t)) \land m,u) du \right) (1 - p_{it}) \prod_{t < \tau} p_{i \tau}$$

$$= \mathbb{E} \sum_{t} \left[ c_{it}^{(m)} \exp(k(t,m)) p_{it} + F_{it}^{(m)}(s(t)) \exp(k((t+s(t)),m))(1 - p_{it}) \right] \prod_{t < \tau} p_{i \tau}$$

(1)

(cf. Fooladi et al. (1997), formula (1)) where the summation is over all $t \in \{1,2,\ldots,T\}$, the multiplication over all $r \in \{1,2,\ldots,t-1\}$ and $\prod_{r < t} p_{i \tau} = 1$. Hence

$$V_i(0) = \sum_{t} \left[ c_{it}^{(m)} p_{it} + F_{it}^{(m)}(s(t))(1 - p_{it}) \right] \prod_{t < \tau} p_{i \tau}$$

(2)

is the expected time-$m$ present value of $i$th bond when $k(t,m) = 0$ for all $t$.

A portfolio $q = (q_1,\ldots,q_n)$ is called a feasible portfolio if $\sum_{i=1}^{n} q_i V_i(0) = L$ where $q_i \geq 0$ for every $i$ and $L$ means a liability due at time $m$. 
Main result

Denote by $\sum_{i=1}^{n} q_i V_i(k)$ the expected value of a defaultable bond portfolio $q$ at time $m$ under the assumption that the shock $k$ appeared. Let the term structure of interest rates $\{f(t,s), s \geq t \geq 0\}$ be a random field holding under Assumption 1.

**Assumption 1.** $\sup_{s \geq t} |f(t,s) - f(0,s) - \delta(t)| \leq \lambda$ for all $0 \leq t \leq m$, where $\{\delta(t), t \geq 0\}$ is a stochastic process with mean $\mu(t)$ and $0 \leq \lambda < \infty$ is a fixed number.

How $\delta$ is specified depends on investor’s preferences. We put

$$\delta(t) = \frac{1}{T-t} \int_{t}^{T} (f(t,s) - f(0,s)) ds,$$

which states that $\delta(t)$ is an average value of shock on $[t,T]$.

Define the following class of shocks:

$$K = \left\{ k : k(t,s) = \int_{t}^{s} (f(t \wedge s, u) - f(0, u)) du, s, t \geq 0 \right\}.$$

Our aim is to choose a feasible portfolio which maximize

$$\inf_{k \in K} \frac{1}{T} \sum_{i=1}^{n} q_i V_i(k).$$

Given bond-pricing equation (1), we propose a new definition of a risk adjusted duration measure.

**Definition 1.** The modified duration of the portfolio $q=(q_1,q_2,\ldots,q_n)$ adjusted for default risk is called a quantity defined by
It is clear that if $\mu(t)$ is constant then

$$D(q) = \frac{1}{L} \sum_{i=1}^{n} \frac{q_i \sum_{t} [\mu(t \wedge m)c_{it}^{(m)}p_{it} + (t + s(t))\mu((t + s(t)) \wedge m)F_{it}^{(m)}(s(t))(1 - p_{it})] \prod_{r<t} p_{rt}}{\sum_{i=1}^{n} q_i \sum_{t} [\mu(t \wedge m)c_{it}^{(m)}p_{it} + \mu((t + s(t)) \wedge m)F_{it}^{(m)}(s(t))(1 - p_{it})] \prod_{r<t} p_{rt}}. \quad (3)$$

since

$$L = \sum_{i=1}^{n} q_i V_i(0) = \sum_{i=1}^{n} q_i \sum_{t} [c_{it}^{(m)}p_{it} + F_{it}^{(m)}(s(t))(1 - p_{it})] \prod_{r<t} p_{rt}. \quad (4)$$

Obviously, if the portfolio is default-free (i.e. $p_{it} = 1$ for all $i, t$) then

$$D(q) = \frac{1}{L} \sum_{i=1}^{n} q_i \sum_{t} c_{it}^{(m)} \quad (5)$$

is the Fisher and Weil duration of portfolio (1971) unadjusted for risk and calculated with respect to changes in the risky term structure.

**Definition 2.** By the modified M-Absolute of the portfolio $q = (q_1, q_2, \ldots, q_n)$ we mean

$$M(q) = \frac{1}{L} \sum_{i=1}^{n} q_i \sum_{t} [m - t\omega_{it}^{(m)}p_{it} + (m - t - s(t))\mu_{it}^{(m)}(s(t))(1 - p_{it})] \prod_{r<t} p_{rt}. \quad (6)$$

For default free bond portfolio (6) becomes

$$M(q) = \frac{1}{L} \sum_{i=1}^{n} q_i \sum_{t} [m - t\omega_{it}^{(m)}], \quad (7)$$

which is the Nawalkha and Chambers (1996) dispersion measure, called M-Absolute.
**Theorem 1.** Under the conditions stated above,

\[
\inf_{k \in K} \left( \frac{1}{L} \sum_{i=1}^{n} q_i V_i(k) \right) \geq \exp \left( -\lambda M(q) + (m - D(q)) \frac{L_{\mu}}{L} \right),
\]

(8)

where \( L_{\mu} = \sum_{i=1}^{n} q_i \sum_{t} \left[ \mu(t \wedge m) e_{it}^{(m)} p_{it} + \mu((t + s(t)) \wedge m) F_{it}^{(m)} (s(t))(1 - p_{it}) \right] \prod_{\tau < t} p_{it} \).

**Proof.** See Appendix.

In consequence, the immunization strategy consists in selecting a feasible portfolio \( q \) which maximizes right-hand side of inequality (8).

**Corollary.** Let put \( \mu(t) = \mu \) in Assumption 1. Then

\[
\inf_{k \in K} \left( \frac{1}{L} \sum_{i=1}^{n} q_i V_i(k) \right) \geq \exp \left( -\lambda M(q) + (m - D(q)) \mu \right)
\]

where \( D(q), M(q) \) are given in (4) and (7), respectively.

The immunization strategy occurs as a choice of a portfolio which

maximizes \( (m - D(q)) \mu - \lambda M(q) \) subject to \( L = \sum_{i=1}^{n} q_i V_i(0) \).

**Remark 1.** In the case when \( \mu \) is unknown then an investor should find a feasible portfolio which

minimizes \( M(q) \) subject to \( D(q) = m \).

Additional if \( \lambda \equiv 0 \) we obtain that duration-matching portfolio is perfectly immunized with respect to the expected cash flow.
Remark 2. The detailed analysis of the proof shows that Theorem 1 holds under the following weaker

Assumption 1’. \( y(t,m) \geq y_0(t,m) + \delta(t) - \lambda \) for \( t \leq m \) and \( y(m,t) \leq y_0(t,m) + \delta(t) + \lambda \) for \( t > m \), where

\[
y(t,s) = \frac{1}{s-t} \int_t^s f(t,s) ds \quad \text{for} \quad t < s \quad \text{and} \quad y_0(t,m) = \frac{1}{m-t} \int_t^m f(0,s) ds.
\]

Obviously, \( y(t,s) \) is a yield to maturity over the time interval \((t,s)\) and \( y_0(t,m) \) denotes a yield to maturity if the change in the term structure does not appear.

Example. In order to present the above results in reality consider a situation when an investor owes $1,000,000 and has to pay it off in two years from today. Additionally, he is interested in purchasing defaultable bonds to discharge his liability. For simplicity, assume that there are two types of bonds of $1000-face value on the market. The first matures in 1 year whereas the other in 3 years. The annual coupon rates are 10% and 6%, respectively. The coupons paid before second year will be reinvested by purchasing strip bonds. It is predicted that the firm issuing second type of bonds may default with probability 0.01 during second year. In this case, there will be a delayed payment of each bond of $80 at time 4. Under these circumstances an investor’s aim is to acquire a portfolio consisting of 1-year default-free bond and 3-year bond with default risk which will give him the lowest of the highest average losses over the planning horizon. The vector \( q = (q_1, q_2) \) gives us the number of \( i \)th bond units \( q_i \) that the investor bought at time 0 and represents his portfolio. Because a bond portfolio present value is a random variable, Corollary to Theorem 1 helps in

finding a portfolio \( q = (q_1, q_2) \) which maximizes

\[
(m - D(q))u - \lambda M(q) \quad \text{subject to} \quad L = q_1V_1(0) + q_2V_2(0).
\]
To be able to compute the number of the first and second bond units, assume a flat term structure with the yield to maturity $4\%$ and put market parameters basing on detailed observations $\mu(t) = \mu = 0.2\%$, $\lambda = 3\%$. Denote 2-year investment horizon by $m$ and the investor’s financial liability of $1,000,000$ by $L$. It is easy to check that the time-2 present value of 1-year bond is $V_1(0) = 1144$ and of 3-year bond $V_2(0) = 1141.77$. By (4) we get the duration of a portfolio $D(q) = \frac{1144q_1 + 3241.85q_2}{10^6}$, while the portfolio dispersion measure by (6) is equal to $M(q) = \frac{1144q_1 + 1083q_2}{10^6}$. Then the optimal solution is $q = (874.126, 0)$ which means that an investor should buy 874.126 units of 1-year bond to get the minimal value of the highest average losses at time 2. However, the above approach may be applied if an investor can estimate parameter $\mu$. Otherwise, he should choose a portfolio which

\[ \text{minimizes } M(q) \text{ subject to } D(q) = m \]

(see Remark 1). Solving this optimization problem we obtain $q = (398.88, 476.17)$. An investor should buy 398.88 units of 1-year bond and 476.17 of 3-year bond.

Appendix

Proof of Theorem 1. Put

\[ C_0(t, q) = \sum_{s \leq t} \sum_{i=1}^{n} \frac{q_i}{L_0} c^{(m)}_{it} p_{it} \prod_{\tau < t} p_{i\tau}, \]

\[ C_1(t, q) = \sum_{s \leq t} \sum_{i=1}^{n} \frac{q_i}{L_1} c^{(m)}_{it} p_{it} \prod_{\tau < t} p_{i\tau}, \]

where

\[ L_0 = \sum_{i=1}^{n} q_i c^{(m)}_{it} p_{it} \prod_{\tau < t} p_{i\tau}, \]
\[ L_1 = \sum_{i=1}^{n} \sum_{t=1}^{n} q_i F_{t_i}^{(m)}(s(t))(1-p_{it}) \prod_{t < t'} p_{it}. \]

Obviously,
\[ L_0 + L_1 = \sum_{i=1}^{n} q_i V_i(0) = L. \]

Since we exclude short position, functions \( t \rightarrow C_0(t,q) \) and \( t \rightarrow C_1(t,q) \) define probability distributions on \([0,T]\) for any \( q = (q_1,q_2,\ldots,q_n) \). From this and by (1) we get
\begin{align*}
\frac{1}{L} \sum_{i=1}^{n} q_i V_i(k) &= \mathbb{E} \left[ \int_{0}^{T} \exp(k(t,m))dC_0(t,q) \frac{L_0}{L} + \mathbb{E} \left[ \int_{0}^{T} \exp(k(t+s(t),m))dC_1(t,q) \frac{L_1}{L} \right. \right] . \tag{9}
\end{align*}

From Assumption 1 it follows that \( f(t,s) - f(0,s) - \delta(t) \geq -\lambda \) for \( t \leq m \) and \( s \geq t \), thus
\[ k(t,m) - \delta(t)(m-t) = \int_{t}^{m} [f(t,s) - f(0,s) - \delta(t)]ds \geq -\lambda(m-t) \text{ for } t \leq m. \tag{10} \]

Moreover, if \( t > m \) then \( f(m,s) - f(0,s) - \delta(m) \leq \lambda \). Consequently, for \( s \geq m \)
\[ k(t,m) - \delta(m)(m-t) = \int_{t}^{m} [f(m,s) - f(0,s) - \delta(m)]ds \]
\[ = -\int_{m}^{t} [f(m,s) - f(0,s) - \delta(m)] \geq -\lambda(t-m) \text{ for } t > m. \tag{11} \]

From (10), (11) and by the Jensen inequality we have
\begin{align*}
\mathbb{E} \left[ \int_{0}^{T} \exp(k(t,m))dC_0(t,q) \right] &\geq \mathbb{E} \left[ \int_{0}^{T} \exp(\delta(t \wedge m)(m-t) - \lambda |m-t|)dC_0(t,q) \right] \\
&\geq \mathbb{E} \left[ \int_{0}^{T} \exp(\delta(t \wedge m)(m-t)dC_0(t,q) - \lambda |m-t|dC_0(t,q)) \right] \\
&\geq \exp \left( \int_{0}^{T} \mathbb{E} \left[ \delta(t \wedge m)(m-t)dC_0(t,q) - \lambda |m-t|dC_0(t,q) \right] \right). \tag{12}
\end{align*}
Similarly,

\[ \mathbb{E} \int_0^T \exp(k(t + s(t), m))dC_1(t, q) \geq \exp\left( \int_0^T \mathbb{E} \delta((t + s((t)) \wedge m)(m - t - s(t))dC_1(t, q) \right) \]

\[ - \lambda \int_0^T |m - t - s(t)|dC_1(t, q) \right). \quad (13) \]

From (9), (12) and (13) we get

\[ \frac{1}{L} \sum_{i=1}^n q_i V_i(k) \geq \frac{L_0}{L} \exp\left( \int_0^T \mu((m - t))dC_0(t, q) - \lambda \int_0^T |m - t|dC_0(t, q) \right) \]

\[ + \frac{L_1}{L} \exp\left( \int_0^T \mu((t + s(t)) \wedge m)(m - t - s(t))dC_1(t, q) - \lambda \int_0^T |m - t - s(t)|dC_1(t, q) \right). \]

Since \( L_0 + L_1 = L \), applying the Jensen inequality again, we obtain

\[ \frac{1}{L} \sum_{i=1}^n q_i V_i(k) \]

\[ \geq \exp\left( \int_0^T \mu((m - t))dC_0(t, q) \frac{L_0}{L} + \int_0^T \mu((t + s(t)) \wedge m)(m - t - s(t))dC_1(t, q) \frac{L_1}{L} \right) \]

\[ - \lambda \int_0^T |m - t|dC_0(t, q) \frac{L_0}{L} - \lambda \int_0^T |m - t - s(t)|dC_1(t, q) \frac{L_1}{L} \right). \quad (14) \]

From (3), (6) and (14) we conclude that

\[ \inf_k \frac{1}{L} \sum_{i=1}^n q_i V_i(k) \geq \exp\left( (m - D(q)) \frac{L_0}{L} - \lambda M(q) \right), \]

which completes the proof. \( \square \)

References


